

Wind-driven ocean circulation—Part 1. Linear theory and perturbation analysis

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Abstract—The linear model of wind-driven circulation with bottom friction is solved by boundary-layer methods and a perturbation solution of the non-linear model is also presented. The results are discussed in detail to set the stage for the discussion of the results of the numerical integrations of the non-linear system of Part 2.

1. INTRODUCTION

THE PROBLEM of determining the response of the ocean to the action of a wind-stress at the surface has been investigated by many scientists in the past seventy years. Nearly all of the models for large scale wind-driven ocean circulation involve the simplification to a system which, for all practical purposes, is barotropic or homogeneous. The hope in such an investigation is the the principal features of wind-driven ocean circulation can be reproduced or at least approximated.

This “equivalent” barotropic model is constructed so as to incorporate in some sense the facts that the ocean is stratified and that it lies on a rotating sphere. It suffices for our purposes to note that the stratification causes the vertical component of the Coriolis parameter to be much more important than the horizontal component so that the latter is neglected. Then the transformation of the equations to the β -plane may be made in a straightforward manner and the mathematical problem for the vertically averaged velocity is expressible as the vorticity equation for a homogeneous, two-dimensional ocean confined between given boundaries where the stream-function vanishes.

Even though the model is a very crude description of wind-driven ocean circulation (it leaves out the effects of topographic and baroclinic processes), it is a worthwhile model to pursue—at least up to a point. STOMMEL's (1948) elegantly simple demonstration that westward intensification can be directly traced to the β -effect in the linear model bears witness to the usefulness of this simple model. How much more the complete model contains has not been made clear principally because of the difficulties presented by the non-linearities of the equations.

And yet the model itself is sufficiently simple so that we should be able to extract essentially all of the information contained in it. For example, consider the response of a two-dimensional ocean in a basin of simple geometric shape on a β -plane and subject to a steady wind-stress. The equations for this system contain only two parameters, one a measure of friction, the other a measure of non-linearity. With modern electronic computers we have the means for solving this system for different relative values of the parameters. We can learn not only how the flow in such a

model reflects the behaviour of the real ocean but, more important still, we can find out what features the model fails to predict. We can then get on to the more important task of constructing a more complete model, incorporating those features of the present model which are found to be realistic.

This paper contains a systematic investigation of the barotropic model of wind driven ocean circulation. Part 1 includes boundary-layer analyses of the linear problem with bottom friction and of the first perturbation correction by non-linear terms. Emphasis is placed on those concepts and results of the linear and quasi-linear problems which prove to be useful in understanding and even predicting the results of the non-linear system. Most of the results in this paper are explicitly or implicitly contained in the papers by STOMMEL (1948) and by MUNK, GROVES and CARRIER (1950). Our emphasis is different but then so is our vantage point.

Part 2 contains numerical integrations of the finite difference formulation of the differential equations.

2. GOVERNING EQUATIONS

In accordance with the remarks in the introduction we shall make sparing but critical use of the fact that the ocean is a shallow layer of stratified water on the rotating earth. In transforming the equations of motion to the β -plane we note that the stratification of the ocean causes the direction of the local normal to take precedence over the direction given by the axis of rotation. For a truly homogeneous body of water the direction of the axis of rotation would be the critical one*. Because of the stratification the locally horizontal component of the Coriolis parameter can be neglected and, if attention is riveted on a portion of the ocean which is confined to latitudes which are not too far north, the principal manifestation of the sphericity of the earth is reflected only in the variation of the locally normal component of the Coriolis parameter with latitude. In what follows the equations are written in terms of vertically averaged velocities.

With these simplifications and approximations the equations of motion for an ocean basin confined between two level surfaces and subject to a wind-stress at the upper surface are given by

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + f \mathbf{x} \mathbf{v} = -\frac{1}{\rho} \nabla p - K \mathbf{v} + \frac{\boldsymbol{\tau}}{D} \quad (2.1)$$

$$\nabla \cdot \mathbf{v} = 0 \quad (2.2)$$

where $\mathbf{v} = (u, v)$ is the horizontal velocity vector with u taken positive toward the east (x) and v taken positive toward the north (y). The vertical component of the Coriolis parameter, $f = f_0 + \beta y$, is the familiar linearized form used on the β -plane and reflects the variation with latitude. Here, $f_0 = 2\Omega \sin \phi_0$ and $\beta = (2\Omega/a) \cos \phi_0$, where Ω is the rate of rotation of the earth in terms of radians per second, a is the radius of the earth, and ϕ_0 is the latitude about which spherical surface is transformed to the β -plane.

Frictional dissipation in the present model is confined to simple bottom friction with a coefficient, K . It can be derived as an approximate form of the frictional process either by treating friction in terms of a simple drag law (in which case K

*Cf. VERONIS (1963a, b) for a detailed discussion of these points.

should depend on the velocity) or by means of a bottom Ekman layer. In our analysis K is taken as a constant. The wind-stress, $\tau = (\tau^x, \tau^y)$, enters as the surface value of the vertical stress term when the equations are integrated in the vertical and the depth, D , enters into the denominator because of the vertical averaging. Equation (2.2) is the horizontally non-divergent form of the continuity equation.

Equations (2.1) and (2.2) are the basic equations of our model. There is no obvious way to derive them as a first-order set of equations for a baroclinic ocean with bottom topography. For this reason it may be preferable to look upon this model as a very simple model in which a restricted number of processes act and to study the interaction of these processes. In terms of ocean circulation we are as interested in learning about the limitations of the model as we are about the successes. In any event in the subsequent development we shall refer to the model as a two-dimensional flow of a homogeneous fluid on the β -plane even though it is in some sense meant to represent a vertical average of a three-dimensional inhomogeneous flow.

Because the flow is two-dimensional we introduce a stream function, ψ , such that the velocities are defined as

$$u = -\frac{\partial\psi}{\partial y}, \quad v = \frac{\partial\psi}{\partial x} \quad (2.3)$$

and the continuity equation (2.2) is automatically satisfied. The pressure can be eliminated from the system by taking the curl of equations (2.1) and the resulting equation for the vertical component of vorticity, ζ , is

$$\frac{\partial\zeta}{\partial t} + \mathbf{v} \cdot \nabla\zeta + \beta v = -K\zeta + \frac{1}{D} \left(\frac{\partial\tau^y}{\partial x} - \frac{\partial\tau^x}{\partial y} \right) \quad (2.4)$$

where $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ is the vorticity. Using (2.3), we can write

$$\zeta = \nabla^2 \psi \quad (2.5)$$

If the curl of the wind-stress vanishes at the northern and southern boundaries of a region, it has been a customary procedure in the past (and can be justified for the linear boundary-layer type of problem) to treat the region as a closed basin, isolated from the rest of the ocean. We proceed with this assumption (and shall justify it) for the linear model but shall relax the condition for the non-linear system. For the present it suffices to note that the basin is square in shape with boundaries at $x = 0$, πL and $y = 0$, πL and the boundary condition is consequently

$$\psi = 0 \text{ on } x = 0, \pi L \text{ and } y = 0, \pi L \quad (2.6)$$

Most of the detailed calculations presented later have been carried out for a steady wind-stress distribution of the form*

$$\tau^x = -\frac{W}{2} \sin \frac{x}{L} \cos \frac{y}{L}, \quad \tau^y = \frac{W}{2} \cos \frac{x}{L} \sin \frac{y}{L} \quad (2.7)$$

where $\frac{W}{2}$ is the amplitude of each component of wind-stress. The curl of the wind-stress [the last term in equation (2.4)] is then

*N.B. Only the *curl* of the wind-stress enters into the vorticity equations, and many different forms of the wind-stress components could be used to yield the form of $\text{curl } \tau$ given in (2.8). We choose the form which has no divergence in order to simplify the discussion of the pressure field.

$$\frac{1}{D} \left(\frac{\partial \tau^y}{\partial x} - \frac{\partial \tau^x}{\partial y} \right) = - \frac{W}{DL} \sin \frac{x}{L} \sin \frac{y}{L}. \quad (2.8)$$

It is convenient to non-dimensionalize the system so as to introduce a minimum number of parameters and this is done by taking

$$\frac{\partial}{\partial x} = \frac{1}{L} \frac{\partial}{\partial x'}, \quad \frac{\partial}{\partial y} = \frac{1}{L} \frac{\partial}{\partial y'}, \quad \psi = \frac{W}{D\beta} \psi', \quad \frac{\partial}{\partial t} = \beta L \frac{\partial}{\partial t'}. \quad (2.9)$$

Then we transform the vorticity equation (2.4) into

$$\frac{\partial}{\partial t} + RJ(\psi, \zeta) + \frac{\partial \psi}{\partial x} = -\epsilon \zeta + \text{curl } \tau \quad (2.10)$$

where the variables in (2.10) are non-dimensionalized according to (2.9) and the primes have been dropped. The term $\text{curl } \tau$ now contains only the form of the curl of the wind-stress since the amplitude has been divided out.

The parameters R and ϵ are defined as

$$R = \frac{W}{D\beta^2 L^3}, \quad \epsilon = \frac{K}{\beta L} \quad (2.11)$$

and represent respectively the magnitude of non-linear and frictional processes.

The boundary conditions are

$$\psi = 0 \text{ on } x = 0, \pi \text{ and } y = 0, \pi. \quad (2.12)$$

Hence, equations (2.5), (2.10), (2.11) and 2.12) define the boundary value problem that must be solved.

Although the stream function, hence the velocities, can be derived from the four equations cited it is very enlightening to look at the pressure distribution as well. To this end we non-dimensionalize the equations (2.1) using (2.9) and the definition

$$p = \frac{WL\rho}{D} p' \quad (2.13)$$

Then in non-dimensional form the components of equations (2.1) become

$$\frac{\partial u}{\partial t} + R \mathbf{v} \cdot \nabla u - fv = - \frac{\partial p}{\partial x} - \epsilon u + \frac{\tau^x}{2} \quad (2.14)$$

$$\frac{\partial v}{\partial t} + R \mathbf{v} \cdot \nabla v + fu = - \frac{\partial p}{\partial y} - \epsilon v + \frac{\tau^y}{2} \quad (2.15)$$

where

$$f = \frac{a}{L} \tan \phi_0 + y \quad (2.16)$$

and where a is the radius of the earth and ϕ_0 is the latitude about which the spherical surface is transformed onto the β -plane.

For numerical calculations we shall use the following values for the dimensional parameters :

$$\begin{aligned} \Omega &= 0.73 \times 10^{-4} \text{ sec}^{-1} \\ L &= 2 \times 10^8 \text{ cm} \\ a &= 6.4 \times 10^8 \text{ cm} \\ \phi_0 &= 30^\circ. \end{aligned}$$

We can then derive

$$\beta = 2 \times 10^{-13} \text{ cm}^{-1} \text{ sec}^{-1}$$

$$\frac{a}{L} \tan \phi_0 = 1.85$$

The values of ϵ and R will be varied over large ranges.

The linear, steady problem ($R = 0, \partial/\partial t \equiv 0$) was first solved by STOMMEL (1948). It is solved again in the next section with boundary-layer methods so that boundary-layer ideas are brought out and so that the pressure distribution for the linear problem can be discussed.

3. LINEAR MODEL

In order to bring out some of the important ideas and processes which will be useful for an understanding of the non-linear analysis, we consider here the linearized steady system which was originally solved by STOMMEL (1948).

Let $R \equiv 0, \frac{\partial}{\partial t} \equiv 0$ so that equation (2.10) becomes

$$\epsilon \nabla^2 \psi + \psi_x = \text{curl } \tau = -\sin x \sin y \tag{3.1}$$

where subscript x corresponds to $\frac{\partial}{\partial x}$. The boundary conditions are given by (2.12).

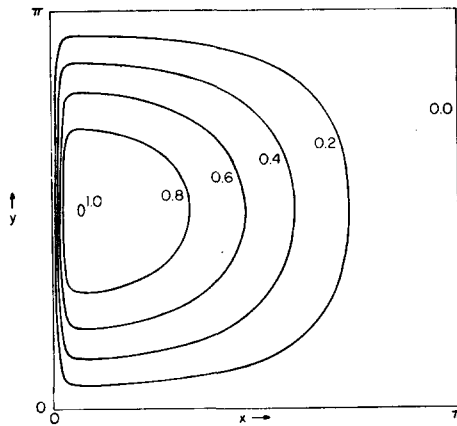


FIG. 1. Contour lines of the normalized stream function, ψ/ψ_{\max} , are shown for values of ψ/ψ_{\max} ranging from 0 to 1.0 in intervals of 0.2. The graph corresponds to equation (3.2) with $\epsilon = 0.05$. The flow is clockwise.

The formal solution to the problem is easily derived. We note that $\psi \sim \sin y$ satisfies both the equation and the y boundary conditions and a one-dimensional problem results. The complete solution is

$$\psi = \frac{1}{1+4\epsilon^2} \left\{ 2\epsilon \sin x + \cos x + \frac{1}{e^{\pi D_1} - e^{\pi D_2}} [(1+e^{\pi D_2})e^{D_1 x} - (1+e^{\pi D_1})e^{D_2 x}] \right\} \sin y \tag{3.2}$$

where $D_1 = \frac{-1 - \sqrt{1+4\epsilon^2}}{2\epsilon}$, $D_2 = \frac{-1 + \sqrt{1+4\epsilon^2}}{2\epsilon}$.

The solution (Fig. 1) shows a southward drift throughout the ocean except for a thin boundary layer along the western coast. The stream function is symmetric in y

and the flow is directed eastward in the northern half and westward in the southern half of the basin. Before discussing the solution, we shall derive it by the approximate but more instructive, boundary-layer method.

(a) *Boundary-layer solution*

In the non-dimensionalization leading to equation (2.10) the stream function was scaled so that the β -term and the driving term have coefficients equal to one. In this linear problem (3.1) this means that only the parameter, $\epsilon = \frac{K}{\beta L}$, enters explicitly and it multiplies the frictional term. The length has been scaled with L so that x and y each range from 0 to π . Let us now assume that the lengths and ψ have been scaled so that the stream function and its derivatives are $O(1)$. Then if $\epsilon \ll 1$, we neglect the frictional term and an approximate form of (3.1) is*

$$v = \psi_x = -\sin x \sin y \quad (3.3)$$

with solution

$$\psi = \cos x \sin y + g(y) \quad (3.4)$$

There is obviously not enough arbitrariness in (3.4) to satisfy all of the boundary conditions. If $g(y) \propto \sin y$, we can cause ψ to vanish at $y = 0$ and π . However, ψ cannot be made to vanish at both $x = 0$ and π . Hence we must relax our assumption that ψ and all of its derivatives are $O(1)$ and thereby allow the frictional term to become important somewhere, presumably near one of the boundaries.

Proceeding with the boundary-layer method, we now assume that in some region $\frac{\partial}{\partial x} = \epsilon^n \frac{\partial}{\partial \xi}$. Then (3.1) becomes

$$\epsilon^{1+2n} \psi_{\xi\xi} + \epsilon \psi_{yy} + \epsilon^n \psi_{\xi} = -\sin x \sin y \quad (3.5)$$

It is clear that $n < 0$ otherwise the left-hand side is at most $O(\epsilon)$ and no term can balance the right-hand side. If $n < -1$, the first term dominates the entire equation and the solution does not have a boundary layer. We choose n such that the first and third terms balance. This yields $n = -1$ and (3.5) becomes

$$\psi_{\xi\xi} + \psi_{\xi} = O(\epsilon) \quad (3.6)$$

with the solution

$$\psi = h(y) e^{-\xi} + k(y) = h(y) e^{-x/\epsilon} + k(y) \quad (3.7)$$

The total approximate solution to the problem consists of the sum of (3.4) and (3.7) and is

$$\psi = \cos x \sin y + h(y) e^{-x/\epsilon} + l(y) \quad (3.8)$$

If $h(y)$ and $l(y)$ are chosen so as to make $\psi = 0$ on $x = 0, \pi$, we have

$$\psi = (1 + \cos x - 2e^{-x/\epsilon}) \sin y \quad (3.9)$$

The solution satisfies the equation to $O(\epsilon)$ and the boundary conditions to $O(e^{-\pi/\epsilon})$. It should be noted that the exponential portion of the solution becomes small for

*This approximate solution is the basis for taking the boundaries of the basin as the lines along which the curl of the wind-stress vanishes. The meridional velocity is directly related to the curl of the wind-stress in this so-called SVERDRUP (1947) solution.

$x > \epsilon$ so that except for a thin region near $x = 0$ the solution is given by the first two terms in the parenthesis of (3.9)

The velocity fields are easily derived from the definition (2.3)

$$v = \psi_x = (-\sin x + 2/\epsilon e^{-x/\epsilon}) \sin y \quad (3.10)$$

$$u = -\psi_y = (2e^{-x/\epsilon} - 1 - \cos x) \cos y \quad (3.11)$$

It is easily seen that with $\epsilon \ll 1$, equation (3.2) reduces to (3.9) because $D_1 \simeq -1/\epsilon$ and $D_2 \simeq \epsilon$ so that $e^{\pi D_1} \ll 1$ and $e^{\pi D_2} \simeq 1$.

(b) Interior flow

Away from the western boundary ($x = 0$) equation (3.3) is valid. Thus there is a southward flow throughout the interior part of the ocean. The significance of this flow is more readily appreciated if we note that it corresponds to a balance between the β -term and the driving term in the vorticity equation. The term, βv , may be written as $v \cdot \nabla f$ or as df/dt . Hence, we have the following approximate balance throughout the interior :

$$\frac{df}{dt} = \frac{\text{curl } \tau}{D} \quad (3.12)$$

In the linear problem the local vorticity of a particle is small relative to the planetary vorticity. Hence, if the wind-stress transmits negative (say) vorticity into the ocean at a given rate, a fluid particle will flow towards lower latitudes with a velocity such that the rate at which its vorticity becomes smaller (smaller because of the smaller value of f at the latitude into which the particles moves) just balances the rate of input of negative vorticity by the wind. The zonal component of velocity has no effect on the vorticity in this linear frictionless region. However, it must be consistent with the driving force and the pressure field and the direction of the interior flow determines both the side of the ocean at which the boundary layer forms and the sign of the vorticity in the boundary layer. We postpone the discussion of the zonal velocity until we consider the pressure distribution.

We must keep in mind the fact that the foregoing results apply to the interior region of a linear system which has boundary-layer characteristics. Non-linear inertial effects can alter this simple picture.

(c) The boundary layer

The boundary layer which provides the northward flow in the system is thin. If we arbitrarily take its characteristic thickness as the length at which the northward velocity falls to, say, e^{-1} of its maximum value at the boundary, we can then state that the boundary-layer thickness is ϵ/π times the scale of the basin*. It is natural to identify the western boundary layer with the Gulf Stream. We turn to observation and find the ratio of Gulf Stream thickness to ocean width to be approximately 0.01. Hence, if our simple linear model is appropriate and if the Gulf Stream is frictionally controlled, then we would have $\epsilon/\pi \simeq 0.01$. This puts an upper bound on ϵ (and therefore K) whether non-linear effects enter or not, because a larger value of ϵ would cause the smallest scale of motion to be correspondingly larger.

*N.B. $\epsilon = K/\beta L$ so that the boundary-layer thickness is K/β divided by the width of the ocean. The factor, π , enters explicitly only because of the way that we have scaled the x co-ordinate.

We note also that the boundary-layer thickness as it is defined above can be derived without solving the entire problem because it arises by the balance of the two terms, $\epsilon\psi_{xx}$ and ψ_x , in the boundary layer. It automatically follows that $\partial/\partial x \sim \epsilon^{-1}$ and this sets the scale.

Consider next the vorticity in the boundary layer. Since $\zeta = \nabla^2 \psi$ and since $\psi_{xx} \gg \psi_{yy}$, we have from (3.10)

$$\zeta = v_x = \psi_{xx} = - \left(\cos x + \frac{2}{\epsilon^2} e^{-x/\epsilon} \right) \sin y \quad (3.13)$$

or approximately

$$\zeta \simeq - \frac{2}{\epsilon^2} e^{-x/\epsilon} \sin y \quad (3.14)$$

Thus the vorticity is negative and very large in the boundary layer and the largest absolute value occurs at mid-latitude.

We can, therefore, describe the overall balance of vorticity as follows: In the interior the wind-stress transmits negative vorticity into the ocean at a certain rate and fluid particles flow southward with a velocity which changes their latitude (and hence their planetary vorticity) at the same rate. The fluid comes into the thin boundary layer and a large shear (consequently, a large local vorticity) is generated by the crowding together of the streamlines. As the streamlines turn northward, the planetary vorticity of the particles is increased at a rate which is balanced by the rate of frictional dissipation of negative vorticity. The *total amount* of vorticity which a particle must lose is fixed by the amount of negative vorticity transmitted by the wind-stress in the interior. However, the *rate* at which the vorticity is dissipated is determined by the magnitude of the frictional parameter. The vorticity reaches a maximum absolute value at mid-latitude because up to mid-latitude additional fluid of negative vorticity enters from the interior. North of the mid-latitude no additional fluid enters the boundary layer and, when viscosity has dissipated the vorticity which a fluid particle has acquired in the interior, the particle leaves the boundary layer to begin the cycle over again.

It is important to note that the path of a particle is symmetric about the mid-latitude, that incoming fluid increases the magnitude of the local vorticity in the boundary layer, and that a particle can leave the boundary layer only when it has lost the necessary amount of vorticity through dissipation.

(d) *The pressure field*

From (2.7), (2.14) and (2.15) we see that the appropriate equations of motion for the linear steady problem are

$$-fv = -p_x - \epsilon u - \frac{1}{2} \sin x \cos y \quad (3.15)$$

$$fu = -p_y - \epsilon v + \frac{1}{2} \cos x \sin y \quad (3.16)$$

Consistent with the boundary-layer argument we can neglect ϵu everywhere (even in the boundary layer since u does not become large). Then it is a simple matter to integrate (3.15) to get

$$-f\psi = -p + \frac{1}{2} \cos x \cos y + g(y) \quad (3.17)$$

An integration of (3.16) yields

$$-f\psi = -p + \cos y + \frac{1}{2} \cos x \cos y + h(x) \quad (3.18)$$

where we have used (3.10) for v . Hence,

$$p = f\psi + \left(\frac{1}{2} \cos x + 1\right) \cos y + \text{const} \quad (3.19)$$

The solution is shown in Fig. 2. We see that the lowest pressures occur in the north-western boundary layer and that there is a high pressure region at mid-latitude just outside the boundary layer. In fact, the pressure and stream function fields roughly coincide. The flow is, therefore, nearly geostrophic. This is reflected in the relatively smaller separation of the isobars in the northern half-basin where the variable Coriolis force is larger.

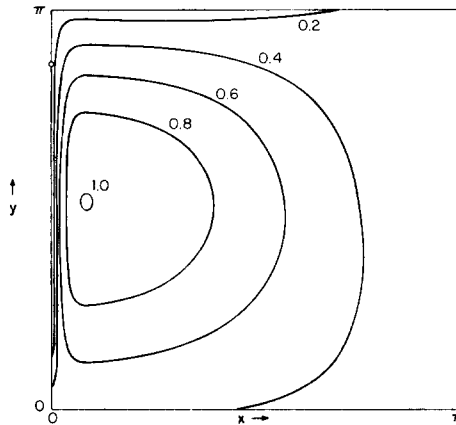


FIG. 2. Contour lines of the normalized pressure field, p/p_{\max} , are shown for values of p/p_{\max} ranging from 0 to 1.0 in intervals of 0.2. The small circle on the western boundary marks the point of minimum pressure ($p = 0$).

The deviation of the interior flow from geostrophy is easily understood if we note that in the interior the equations (3.15) and (3.16) can be written as

$$-f(v_e + v_g) = -p_x - \frac{1}{2} \sin x \cos y \quad (3.20)$$

$$f(u_e + v_g) = -p_y + \frac{1}{2} \cos x \sin y \quad (3.21)$$

where we have divided the velocity field into two parts, the geostrophic part, v_g , and the remaining portion, v_e . The geostrophic velocity is balanced by the pressure force. The remaining portion is balanced by the driving force and is simply the Ekman transport, i.e., the wind-stress gives rise to a transport which is directed to the right of the direction of the wind. Since the wind-stress pattern is clockwise and symmetrical about the center, the resulting Ekman transport is everywhere directed roughly toward the middle of the basin. If such a symmetric pattern is superimposed on the isobar pattern shown in Fig. 2, the net flow must have a component up the pressure gradient everywhere except in the vicinity of the high pressure region where the wind-stress gives rise to a flow toward lower pressure.

Within the boundary layer itself frictional processes enter explicitly and the balance is somewhat more complicated. However, the general behaviour can be described as follows: northward flow is geostrophically balanced by the east-west

pressure gradient. Along the boundary the flow is down the pressure gradient up to high latitude because the balance is mostly between the frictional and pressure forces. In the northernmost regions along the western boundary the pressure begins to build up since the northern boundary is a barrier and the flow there is up the pressure gradient. Away from the immediate vicinity of the boundary the flow in the western boundary layer has a component up the pressure gradient.

We see, therefore, that throughout the ocean basin the flow is primarily geostrophic with some cross-isobar flow generated by the wind-stress in the interior (Ekman transport) and by frictional forces in the boundary layer. The northward flow in the western boundary layer is very strongly geostrophic.

4. PERTURBATION TREATMENT OF INERTIAL EFFECTS

We now consider the effects of the non-linear terms on the linear flow described in the previous section. To do this analytically we must use perturbation analysis, i.e., we can treat only the case where the non-linear effects are in some sense small. Since we shall present non-linear solutions of all orders in Part II, we confine our attention here to qualitative effects of non-linearity and postpone the detailed results to Part II.

The dominant non-linear effects will occur in the western boundary layer in the same manner as the dominant frictional effects. Thus, if the amplitude of the stream function is $O(1)$, non-linear behaviour can be expected to manifest itself principally in those regions where the derivatives of the stream function become large.

Let us write

$$\psi = \psi_f + \psi', \quad \zeta = \zeta_f + \zeta' \quad (4.1)$$

where ψ_f and ζ_f correspond to the linear, frictional solution of (2.10) and ψ' and ζ' are perturbations. We substitute (4.1) into (2.10) (with $\partial\zeta/\partial t \equiv 0$) and find for the lowest order system

$$\epsilon\zeta' + \psi_{x'} = -RJ(\psi_f, \zeta_f) = -R \frac{d\zeta_f}{dt} \quad (4.2)$$

By substituting for $\frac{d\zeta_f}{dt}$ from the linear solution and applying boundary-layer stretching for x , we have the system

$$\psi_{\epsilon\xi\xi'} + \psi_{\xi'} \simeq \frac{2R}{\epsilon^2} e^{-\xi} \sin 2y \quad (4.3)$$

where we have kept only the dominant term on the right-hand side. The boundary-layer solution of (4.3) is (to lowest order in ϵ)

$$\psi' \simeq -\frac{2\xi R}{\epsilon^2} e^{-\xi} \sin 2y \quad (4.4)$$

and, if we now write down the total boundary-layer solution up to first order in R , we have

$$\psi \simeq 2(1 - e^{-\xi}) \sin y - \frac{2R}{\epsilon^2} \xi e^{-\xi} \sin 2y \quad (4.5)$$

$$v \simeq \frac{2}{\epsilon} e^{-\xi} \sin y + \frac{2R(\xi - 1)}{\epsilon^3} e^{-\xi} \sin 2y \quad (4.6)$$

$$\zeta \simeq -\frac{2}{\epsilon^2} e^{-\xi} \sin y + \frac{2R(\xi - 2)}{\epsilon^4} e^{-\xi} \sin 2y \quad (4.7)$$

From this perturbation solution we see that the transport, ψ , in the boundary layer is decreased in the southern half-basin and increased in the northern half-basin by the perturbation correction. The net result therefore is to shift the point of maximum transport northward.

Furthermore, in the southern half-basin the northward velocity is decreased for $\xi < 1$ and is increased for $\xi > 1$ by the perturbation solution. The amplitude of vorticity, $|\zeta|$, is decreased for $\xi < 2$ and is increased for $\xi > 2$. Thus in the southern half-basin inertial effects tend to increase the width of the boundary layer and simultaneously to make it smoother by cutting down the large amplitudes of the northward velocity and vorticity.

In the northern half-basin v is increased for $\xi < 1$ and decreased for $\xi > 1$ and $|\zeta|$ is increased for $\xi < 2$ and decreased for $\xi > 2$. Both v and ζ can change sign near the outer regions of the boundary layer. This fact reflects the effect of inertial processes which can cause a particle to overshoot its equilibrium position. When that happens the particle will return to the equilibrium position by reversing its direction of flow and the sign of its vorticity. Thus a countercurrent is generated on the offshore side of the boundary current.

It is important to note that in the northern half-basin inertial effects tend to concentrate the intense northward flow and thereby intensify the boundary-layer structure. Near the outer regions of the boundary layer the flow turns southward. Therefore a double boundary layer begins to develop in the north as a result of inertial effects.

These results shown that non-linearity provides a north-south asymmetry in the transport function and in the structure of the boundary layer. Because the northward transport in this region is highly geostrophic (it continues to be so for the non-linear solutions) a decrease in the mass transport in the south must be accompanied by a decrease in the east-west pressure gradient, and an increase of mass transport in the north must be associated with an increase in $\partial p/\partial x$. Thus there is a redistribution of fluid such that the high pressure region is located farther to the north and west when the flow is non-linear. This shift of the high pressure region serves several purposes. In addition to providing for the local increase in the northward transport, it makes it more difficult for the fluid to leave the boundary layer since the fluid must now flow toward the east into a region of higher pressure. Hence, there is a tendency for the outgoing fluid to concentrate toward the north. This path is made easier because as the high pressure region moves northward the magnitude of $\partial p/\partial y$ in the northernmost regions is also increased and the resulting eastward geostrophic flow is larger. Hence, inertial effects tend to create a boundary layer on the northern boundary. This is consistent with FOFONOFF's (1954) results for a free, inertially-controlled flow.

We note further that, since both the transport, v , and the magnitude, $|\zeta|$, of the vorticity are smaller in incoming regions and larger in outgoing regions than the corresponding quantities for the linear problem, the points of maximum v and $|\zeta|$ must be shifted

to the north. Thus, there is a tendency for the region of incoming flow to be extended northward past mid-latitude. Without a higher-order calculation we cannot tell whether the total amount of transport is larger or smaller than in the linear problem. And finally, since the frictional force is proportional to amplitude, it is clear that the dissipation will be increased in the northern regions and decreased in the southern regions by inertial effects.

The foregoing results tell us much about what can be expected in the non-linear problem. They are especially useful in indicating the types of response which occur in the western boundary layer and in pointing out the relative roles of non-linearity and friction.

Suppose that we had taken a different tack by treating the problem as one in which the boundary layers were controlled principally by inertial terms. In this case the β -term in equation (2.10) would be balanced by the non-linear terms so that we would have a balance of the type

$$R \psi_x \psi_{xx} \sim \psi_x \quad (4.8)$$

If we again assume that the principal variation is introduced by x -derivatives, it is necessary that

$$\frac{\partial}{\partial x} \sim R^{-1/2} \quad (4.9)$$

Then we can talk of an inertial boundary-layer thickness whose scale is $\sqrt{(R)}/\pi$ of that of the basin.

The difficulty with this approach is that friction cannot be introduced as a perturbation because all of the streamlines must pass through a frictional region in order to dissipate the vorticity which has been accumulated in the interior through the action of the wind-stress. Hence, in at least some region of the basin, the frictional processes must be at least as important as the non-linear processes and in that region a perturbation approach breaks down. On the other hand, when inertial effects are treated by a perturbation approach, the distortion of the linear flow can be determined to any desired accuracy provided that R is sufficiently small. Hence, if the magnitudes, \sqrt{R} and ϵ , are used as some kind of measure of the relative roles of non-linearity and friction, it should be possible to treat the problem by perturbation methods as long as $\sqrt{R} < \epsilon$.

Even in the strongly non-linear problem, i.e., when $\sqrt{R} \geq \epsilon$, it may be possible to analyze some regions by means of an inertial boundary-layer treatment. The results in this section show that inertial effects tend to broaden the frictional boundary layer in the south and cause it to become narrower in the north. Hence, when $\sqrt{R} > \epsilon$, the region of maximum dissipation is probably in the north and the region of incoming flow may be amenable to analysis based on an inertial boundary layer. For a meaningful analysis it is necessary to know the extent of the region of incoming flow but that is not possible without explicit consideration of the northern region where friction is important and where the fluid leaves the boundary layer. Indeed, that one can speak of a boundary layer in the north in the same sense that one uses the term for regions of incoming flow is very doubtful.

We note, also, that a necessary condition for the validity of the perturbation solution (4.5) is that $R/\epsilon^2 < 1$. If we recall the derivation of frictional and inertial

boundary layers, we see that the criterion means that the perturbation approach is valid provided that the inertial boundary-layer thickness is somewhat less than the thickness of the frictional boundary layer. We shall refer to this point again in Part 2.

Finally, we note some of the results of this section which are pertinent to the question of separation of the western boundary layer from the coast, a point which has been discussed a great deal in the past ten years by various authors. The presence of a high pressure region just outside the boundary layer when the system is linear must inhibit the flow from separating at mid-latitude. In fact, as we have noted earlier, this high pressure ensures that there will be a strong northward flow at that latitude because the flow is geostrophically balanced.

Inertial effects tend to shift the high pressure region northwestward so that the northward flow is intensified in that region and, if the tendency is continued for larger values of R , an eastward jet, balanced by a strong north-south pressure gradient, will form at the northern boundary. The indications from the present argument are that separation of the western boundary layer cannot occur at mid-latitude or indeed at any latitude other than along the northern boundary. Hence, non-linearity cannot cause separation in the barotropic system (when the depth is constant). If the fluid is stratified, the picture may be completely different, especially if there is not enough lighter, surface water to allow the geostrophically balanced jet to continue northward.

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